

NPT Bound Entanglement- The Problem Revisited

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Abstract

Recently [quant-ph/0608250] again created a lot of interest to prove the existence of bound entangled states with negative partial transpose (NPT) in any $d \times d$ ($d \geq 3$) Hilbert space. However the proof in quant-ph/0608250 is not complete but it shows some interesting properties of the Schmidt rank two states. In this work we are trying to probe the problem in a different angle considering the work by Dür et.al [Phys. Rev. A, 61, 062313(2000)]. We have assumed that the Schmidt rank two states should satisfy some bounds. Under some assumptions with these bounds one could prove the existence of NPT bound entangled states. We particularly discuss the case of two copy undistillability of the conjectured family of NPT states. Obviously the class of NPT bound entangled states belong to the class of conjectured to be bound entangled states by Divincenzo et.al [Phys. Rev. A, 61, 062312(2000)] and by Dür et.al [Phys. Rev. A, 61, 062313(2000)]. However the problem of existence of NPT bound entangled states still remain open.

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1 Introduction

The basic issue on the classification of mixed state entanglement at least on the level of bipartite systems solely depends upon whether there exist bound

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entangled states or not. The existence of PPT-bound (PPT means positive partial transpose) entangled states [1] and also the existence of NPT N -copy undistillable states [2, 3] for every positive integer N naturally indicates there may exist NPT-bound entangled states. Recent work [4] also indicates positively the existence of NPT-bound entangled states. In this work we consider the problem of the existence of NPT-bound entangled states with some assumptions on Schmidt rank two states. We first briefly describe the issue and the importance of the problem.

In recent years it is found that quantum entanglement is an useful resource in performing several tasks in quantum information theory and quantum communication [5]. Maximally entangled states shared between two parties are essential ingredients in this respect [6]. Now due to the interaction with environment, states are in practice found to be mixed. However there is a process called distillation by which we can distill sometimes maximally entangled states out of certain pair of mixed entangled states using only local operation and classical communications (LOCC) [7]. But entanglement of a state is not always sufficient for distillability. The Peres-Horodecki criterion [8] namely the partial transpose corresponding to any bipartite system gave us a necessary condition for distillability of any entangled state. If a bipartite density operator have positive partial transpose then it is not distillable. Further any PPT-state may be classified into two classes, separable and PPT-bound entangled states (bound entangled states means no entanglement can be extracted from them by LOCC, i.e., not distillable). There exist PPT-bound entangled states [1]. But are all NPT- states which are necessarily entangled, distillable [9]? Until this time there is no answer. Independently, Divincenzo et.al [2] and Dür et.al [3] and also Somshubhro et.al [10] gave some evidence for N -copy undistillable states. Watrous [11] further investigated the problem of distillability with large number of copies of some entangled states. Recently the work by Simon [4] indicates there may exist NPT bound entangled states with a large class of state that includes the conjectured family of NPT bound entangled states. Here we show that NPT-bound entangled states exist for any bipartite $d \times d (d \geq 3)$ system if we consider some simple assumptions on Schmidt rank two states. Our approach is based on some bounds of rank two states that are not closed in [3]. Obviously they belong to the classes as suggested earlier. Lastly we show a simple property that satisfied by a class of conjectured bound entangled states.

With the existence of NPT-bound entangled state it is also proved that the distillable entanglement is nonadditive and not convex [12]. It should be noted that by distillable entanglement [7, 13] of a bipartite state we mean how much pure maximally entangled states we can extract asymptotically

by means of LOCC from several copies of that state. Now, by definition of bound entanglement, every bound entangled state, whether NPT or PPT, has zero distillable entanglement. In [12], Shor et. al showed that distillable entanglement of tensor product of two states, one PPT-bound entangled state (formed by pyramid UPB) and another conjectured to be NPT-bound entangled state (which we shall prove really NPT-bound entangled) is non-zero. Which proves the nonadditivity and non-convexity of distillable entanglement. Also, it constitutes another example that PPT-bound entangled states can be used in the activation process [12, 14, 15].

2 The conjectured class of NPT bound entangled states

Before going to discuss our result, we first mention the notion of distillable states on any bipartite system described by the joint Hilbert space $H_A \otimes H_B$. *Definition* [2, 3, 12].– A density matrix ρ is distillable if and only if there exists a positive integer n such that

$$\langle \psi | (\rho^{\otimes n})^{T_A} | \psi \rangle < 0 \quad (1)$$

for any Schmidt rank two state $|\psi\rangle \in (H_A \otimes H_B)^{\otimes n}$, where T_A represents partial transpose with respect to the system A .

Now we consider the key state in $d \times d$ ($d \geq 3$) Werner class that are one copy undistillable [2, 3, 12] and conjectured to be bound entangled.

Theorem/Conjecture.– The state $\rho(\lambda)$ in $d \times d$, ($d \geq 3$) of Werner class represented by

$$\rho(\lambda) = \frac{1}{d(d + \lambda(d - 1))} [I + \lambda \sum_{i,j,i < j} P(|ij\rangle - |ji\rangle)], \quad (2)$$

where $\frac{1}{d-1} < \lambda \leq 1$ is NPT- bound entangled, where, $\{|0\rangle, |1\rangle, |2\rangle \dots\}$ is an orthonormal basis on the Hilbert space $H_A(H_B)$.

For $\lambda = \frac{1}{d-1}$ the state is separable.

The state is one copy undistillable [2, 3, 12]. One has to prove, it is n -copy undistillable for any n .

2.1 Two copy undistillability under some assumptions

To explain the possibility of two copy undistillability of the conjectured class, we consider first the partial transpose of the given state $\rho(\lambda)$. The partial transpose of the state with respect to the system A is,

$$\rho(\lambda)^{T_A} = \frac{1}{d(d + \lambda(d - 1))} [(1 + \lambda)I - \lambda P(\sum_i |ii\rangle)].$$

In the sequel we write $P^+ = P(\sum_i |ii\rangle)$. Before mentioning our basic assumptions which seems to be correct for any Schmidt rank two states, we first note an observation found for a class of Schmidt rank two states.

An Observation.— Since any Schmidt rank two state $|\psi\rangle$ in $(H_A \otimes H_B)^{\otimes 2}$ has expectation value less than two with the operator $I_{AB} \otimes P_{AB}^+$, therefore the rank two states of the form $|\chi_A \phi_B\rangle \otimes |\psi'\rangle_{AB}$, where $|\psi'\rangle_{AB}$ is any Schmidt rank two state in $(H_A \otimes H_B)$, has expectation value less than three with the operator $I_{AB} \otimes P_{AB}^+ + P_{AB}^+ \otimes I_{AB}$.

(in the text we have used multiple copies of operators of system A, B with the same suffix)

With this simple observation on Schmidt rank two states, if someone ask what will be the case for any general Schmidt rank two states in $(H_A \otimes H_B)^{\otimes 2}$? We mention this as our first assumption.

Assumption 1.— For any Schmidt rank two states $|\psi\rangle$ in $(H_A \otimes H_B)^{\otimes 2}$,

$$\langle \psi | (kI_{AB} - P_{AB}^+) \otimes P_{AB}^+ + P_{AB}^+ \otimes (kI_{AB} - P_{AB}^+) | \psi \rangle \leq \max\{2k, 3k - 4\}, \quad (3)$$

where $k > 2$.

Clearly, with this bound it is now easy to check that for any Schmidt rank two state $|\psi\rangle$ in $(H_A \otimes H_B)^{\otimes 2}$,

$$\langle \psi | (\frac{k}{2}I_{AB} - P_{AB}^+)^{\otimes 2} | \psi \rangle \geq \frac{k^2}{4} - \frac{\max\{2k, 3k - 4\}}{2} \geq 0, \text{ if } k \geq 4 \quad (4)$$

Now, putting $k = \frac{2(1+\lambda)}{\lambda}$, we have,

$$\langle \psi | (\rho(\lambda)^{\otimes 2})^{T_A} | \psi \rangle \geq 0, \quad \text{for } \lambda \leq 1$$

i.e., $\rho(\lambda)$ is two copy undistillable.

2.2 n-copy undistillability under some assumptions

Next we consider some assumptions in $(H_A \otimes H_B)^{\otimes n}$, for any $n \geq 2$. We have assumed a sequence of bounds for any Schmidt rank two state $|\psi\rangle$ in $(H_A \otimes H_B)^{\otimes n}$ where $n \geq 2$. These bounds are not present in [3], however from numerical evidences and also analytically for large classes of Schmidt rank two states we found these are true. Until now we find no exceptional cases that violets these bounds.

$$(i). \quad \langle \psi | (I_{AB})^{\otimes(n-1)} \otimes P_{AB}^+ + \dots + P_{AB}^+ \otimes (I_{AB})^{\otimes(n-1)} | \psi \rangle \leq 2(C_1^n - C_1^{n-1}) + C_1^{n-1} = n+1,$$

$$(ii). \quad \langle \psi | (I_{AB})^{\otimes(n-2)} \otimes (P_{AB}^+)^{\otimes 2} + \dots + P_{AB}^+ \otimes (I_{AB})^{\otimes(n-2)} \otimes P_{AB}^+ + \dots$$

$$+ (P_{AB}^+)^{\otimes 2} \otimes (I_{AB})^{\otimes(n-2)} | \psi \rangle \leq 2(C_2^n - C_2^{n-1}) + C_2^{n-1} = 2(C_1^{n-1}) + C_2^{n-1},$$

$$(iii). \quad \langle \psi | (I_{AB})^{\otimes(n-3)} \otimes (P_{AB}^+)^{\otimes 3} + \dots + P_{AB}^+ \otimes (I_{AB})^{\otimes(n-3)} \otimes (P_{AB}^+)^{\otimes 2} + \dots$$

$$+ (P_{AB}^+)^{\otimes 3} \otimes (I_{AB})^{\otimes(n-3)} | \psi \rangle \leq 2(C_3^n - C_3^{n-1}) + C_3^{n-1} = 2(C_2^{n-1}) + C_3^{n-1},$$

Proceeding in this way we have for any $m < n + 1$,

$$(m). \quad \langle \psi | (I_{AB})^{\otimes(n-m)} \otimes (P_{AB}^+)^{\otimes m} + \dots + P_{AB}^+ \otimes (I_{AB})^{\otimes(n-m)} \otimes (P_{AB}^+)^{\otimes m-1} + \dots$$

$$+ (P_{AB}^+)^{\otimes m} \otimes (I_{AB})^{\otimes(n-m)} | \psi \rangle \leq 2(C_m^n - C_m^{n-1}) + C_m^{n-1} = 2(C_{m-1}^{n-1}) + C_m^{n-1},$$

where $C_r^n = \frac{n(n-1)\dots(n-r+1)}{r(r-1)\dots 2.1}$, for any $r \leq n$.

Now consider the set of all Schmidt rank two states $|\psi\rangle$ in $(H_A \otimes H_B)^{\otimes n}$ that attains the optimal values for all the above bounds. The set is not empty. For such a rank two state $|\psi\rangle$ we have,

$$\langle \psi | (\rho(\lambda)^{\otimes n})^{T_A} | \psi \rangle \geq 0. \quad (5)$$

Now we conjecture that the above result satisfied by any rank two state from the optimal set is also satisfied by any rank two state. In other words we want to say that to prove $\rho(\lambda)$ is n -copy undistillable for any n , the proof for optimal class of states is sufficient. However one may construct some composite bounds like equation (3) that would directly prove the n -copy undistillability of the conjectured class of Werner states represented by $\rho(\lambda)$. We have tested for a large class of rank two states that satisfied by equation(5). Also it is interesting to note that if the conjectured class of states are NPT bound entangled states, then the bounds we have assumed must be satisfied for Schmidt rank two states. Next we consider a simple property satisfied by the conjectured class of bound entangled states.

3 A simple property

For simplicity we consider the state $\rho(\lambda)$ for $\lambda = 1$. We denote it by ρ . First, we consider $\rho^{\otimes 2}$. After rewriting it with first two basis elements for system A then for system B , and omitting normalization factor (as it will not alter the trace condition) it looks as follows:

$$\begin{aligned}
\rho^{\otimes 2} &= I \otimes I + \sum_{i,j,m,n,i < j} [P(|minj\rangle - |mjni\rangle) + P(|imjn\rangle - |jmin\rangle)] \\
&\quad + \sum_{i,j,k,l,i < j, k < l} P(|ikjl\rangle - |iljk\rangle - |jkil\rangle + |jlik\rangle) \\
&= I \otimes I + \sum_{i,j,m,i < j} [P(|mimj\rangle - |mjmi\rangle) + P(|imjm\rangle - |jmim\rangle)] \\
&\quad + \sum_{i,j,k,l,i < j, k < l} [P(|ikjl\rangle - |iljk\rangle) + P(|ikjl\rangle - |jkil\rangle) \\
&\quad + P(|ikjl\rangle - |iljk\rangle - |jkil\rangle + |jlik\rangle)],
\end{aligned} \tag{6}$$

where $i, j, k, l, m, n = 0, 1, 2, \dots$

Now it is easy to check that any off-diagonal operator of the form, $|ijkl\rangle\langle mn pq|$, $i, j, k, l, m, n, p, q = 0, 1, 2, \dots$, occurs maximum once or twice on the above expression. After taking partial transpose say with respect to system A it takes the form $|mnkl\rangle\langle ij pq|$, $i, j, k, l, m, n, p, q = 0, 1, 2, \dots$. Also if we take the partial transpose of $\rho^{\otimes 2}$ with respect to system A , then we find in $(\rho^{\otimes 2})^{T_A}$, $P(|ijpq\rangle), P(|mnkl\rangle), i, j, k, l, m, n, p, q = 0, 1, 2, \dots$, occur same times as $|mnkl\rangle\langle ij pq|$, $i, j, k, l, m, n, p, q = 0, 1, 2, \dots$. Therefore, trace with any Schmidt-rank two state $|\psi\rangle$ in the basis we have represented $(\rho^{\otimes 2})^{T_A}$, would be always non-negative.

Next consider $\rho^{\otimes 3}$. It will take the form, if we write first three basis elements for system A and then for system B , as follows:

$$\begin{aligned}
\rho^{\otimes 3} &= I \otimes I \otimes I + \sum_{i,j,m,n,p,q,i < j} [P(|mpinqj\rangle - |mpjnqi\rangle) \\
&\quad + P(|mipnjq\rangle - |mjpn iq\rangle) + P(|impj nq\rangle - |jmpin q\rangle)] \\
&\quad + \sum_{i,j,k,l,m,n,i < j, k < l} [P(|m i k n j l\rangle - |m i l n j k\rangle - |m j k n i l\rangle \\
&\quad + |m j l n i k\rangle) + P(|i m k j n l\rangle - |i m l j n k\rangle - |j m k i n l\rangle + |j m l i n k\rangle) \\
&\quad + P(|i k m j l n\rangle - |i l m j k n\rangle - |j k m i l n\rangle + |j l m i k n\rangle)]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j,k,l,m,n,i < j, k < l, m < n} P(|ikmjln\rangle - |iknjlm\rangle - |ilmjkn\rangle \\
& + |ilnjkm\rangle - |jkmiln\rangle + |jknilm\rangle + |jlmikn\rangle - |jlnikm\rangle) \quad (7)
\end{aligned}$$

Here again if we consider any off-diagonal operator of the form, $|ijklmn\rangle\langle pqrstu|$, $i, j, k, l, m, n, p, q, r, s, t, u = 0, 1, 2, \dots$, occur maximum one or 2^1 or 2^2 times on the above expression. Therefore, with the similar argument above, we find trace with any Schmidt-rank two state $|\psi\rangle$ in the basis we have represented $(\rho^{\otimes 3})^{T_A}$, would be always non-negative.

For $\rho^{\otimes N}$, if we write it similarly as above, we find off-diagonal operator of the form

$$|ijk \cdots N \text{ times } lmn \cdots N \text{ times}\rangle\langle pqr \cdots N \text{ times } stu \cdots N \text{ times}|,$$

$i, j, k, l, m, n, p, q, r, s, t, u = 0, 1, 2, \dots$, will occur maximum one or 2^1 or $2^2 \cdots$ or 2^{N-1} times and in the partial transpose $(\rho^{\otimes N})^{T_A}$, $P(|pqr \cdots N \text{ times } lmn \cdots N \text{ times}\rangle)$, $P(|ijk \cdots N \text{ times } stu \cdots N \text{ times}\rangle)$, occur same number of times as the off-diagonal operator,

$$|pqr \cdots N \text{ times } lmn \cdots N \text{ times}\rangle\langle ijk \cdots N \text{ times } stu \cdots N \text{ times}|.$$

So the trace with any Schmidt rank two state $|\psi\rangle$ in the basis we have considered $(\rho^{\otimes N})^{T_A}$, would be always non-negative. However for any Schmidt rank two state (i.e., in any other basis), we are unable to calculate the trace with $(\rho^{\otimes N})^{T_A}$ for any N , using the property we have found above. The property we have discussed above for ρ , could be easily extended to any $\rho(\lambda)$.

4 Conclusion

To summarize our results, we have revisited the problem of existence of NPT-bound entangled states of any bipartite systems $d \times d$, $d \geq 3$ with some assumptions made on Schmidt rank two states. The key role plays here the bounds that we have assumed for any rank two states. Their proof would readily solve the problem of classification of states at least at bipartite level, i.e., whether a state is either separable or bound entangled (PPT or NPT) or distillable. There are always some confusion regarding distillability when a bipartite state is NPT. However the problem of existence of NPT bound entangled states remains still open.

Comment. We have started the problem since the year, late 2003. Sometimes we felt, we have solved the problem, after that we found the proof is

not complete yet. Recently (last 8-10 months) we found some bounds by which the problem can be solved. We found them through some requirements of some operators to maintain positivity. However that proof is not most general one. Then we look upon the problem in reverse order using those bounds and found some assumptions that we have mentioned in our first version. After observing the paper quant-ph/0608250, we put our work into the net also. Since the assumption made in our first version is strong enough and need not to be satisfied for all rank two states, therefore we dropped this assumption in second version, which is not actually needed for our proof. In this version we have tried to be more explanatory and also discusses the problem in different angles.

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